

# State transition of a non-Ohmic damping system in a corrugated plane

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Anomalous transport of a particle subjected to non-Ohmic damping of the power  $\delta$  in a tilted periodic potential is investigated via Monte Carlo simulation of generalized Langevin equation. It is found that the system exhibits two relative motion modes: the locking state and the running state. Under the surrounding of sub-Ohmic damping ( $0 < \delta < 1$ ), the particle should transfer into a running state from a locking state only when local minima of the potential vanish; hence the particle occurs a synchronization oscillation in its mean displacement and mean square displacement (MSD). In particular, the two motion modes are allowed to coexist in the case of super-Ohmic damping ( $1 < \delta < 2$ ) for moderate driving forces, namely, where exists double centers in the velocity distribution. This induces the particle having faster diffusion, i.e., its MSD reads  $\langle \Delta x^2(t) \rangle = 2D_{\text{eff}}^{(\delta)} t^{\delta_{\text{eff}}}$ . Our result shows that the effective power index  $\delta_{\text{eff}}$  can be enhanced and is a nonmonotonic function of the temperature and the driving force. The mixture effect of the two motion modes also leads to a breakdown of hysteresis loop of the mobility.

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## I. INTRODUCTION

There are many physical situations which can be described by Brownian transport in a tilted periodic potential, for example, Josephson junction [1], charge-density wave [2], superionic conductor [3], rotation of dipoles in external field [4], phase-locking loop [5], diffusion on surface [6], and separation of particles by electrophoresis [7]. The quantitative properties of those systems have been discussed in plenty of literatures [8], such as the dependence of the coherence level of transport on the temperature, driving force and shape of potential [9]; the huge enhancement of the effective diffusion coefficient relative to the free diffusion [10, 11, 12]; the response of output to the noise and signal [13], and so on [14]. Since theoretical tools and numerical algorithms are not sufficient in non-Markovian dynamics with a frequency-dependent non-Ohmic damping, most of the models are established in the Ohmic damping environment. However, the frequency-dependent damping is more general because a large number of stochastic processes fail to be the Markovian dynamics.

Recent studies on anomalous diffusion and transport are mostly limited in the absence of potential, linear force case or the sub-diffusion in a potential [15]. It is worth to point out that the behavior of a particle moving in a periodic potential immersed in the super-Ohmic damping environment might be far more complicated than that in the Ohmic and sub-Ohmic damping cases. In comparison with the previous findings for great enhancement of the effective diffusion coefficient [10, 11, 12] and the hysteresis loop of mobility [8] in the Ohmic damping environment, we will perform a detailed investigation in the

present work on the diffusion and the mobility of a particle subjected to arbitrary non-Ohmic damping in a corrugate plane. This is in terms of an effective algorithm proposed by us [16] to numerically solve a generalized Langevin equation (GLE) with arbitrary damping kernel function and corresponding thermal colored noise.

The paper is organized as follows. In Sec. II, we describe briefly the anomalous transport model by means of the GLE. In Sec. III, the two basic quantities of interest: the generalized effective diffusion coefficient and the fractional mobility are defined; the novel behaviors of diffusion and mobility are shown. Finally, we draw a conclusion of our findings in the section IV.

## II. THE MODEL

We consider a Brownian particle moving in a one-dimensional periodic potential under the influence of non-Ohmic memory friction and a constant external driving force. The dynamics of the particle is governed by the following GLE [17, 18],

$$\begin{aligned} \dot{x}(t) &= v(t), \\ m\dot{v}(t) &= -m \int_0^t \gamma(t-t')v(t')dt' + U'(x) + \sqrt{mk_B T} \xi(t), \end{aligned} \quad (1)$$

where  $k_B$  is the Boltzmann constant,  $T$  is the temperature of the environment,  $\gamma(t)$  is the damping kernel function and related to  $\xi(t)$  through the fluctuation-dissipation theorem [17, 19]

$$\langle \xi(t)\xi(t') \rangle = \gamma(|t-t'|), \quad (2)$$

where  $\xi(t)$  is a zero mean Gaussian colored noise and its spectral density reads

$$\langle |\xi(\omega)|^2 \rangle = 2\gamma_\delta \left( \frac{|\omega|}{\tilde{\omega}} \right)^{\delta-1} f_c \left( \frac{|\omega|}{\omega_c} \right). \quad (3)$$

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The small  $|\omega|$  behavior of  $\langle |\xi(\omega)|^2 \rangle$  is a power-law characterized by the index  $\delta - 1$ . The function  $f_c(|\omega|/\omega_c)$  is a high frequency cutoff function of typical width  $\omega_c$  [20], and  $\tilde{\omega} \ll \omega_c$  denotes a reference frequency allowing for the constant  $m\gamma_\delta$  to have a dimension of viscosity for any  $\delta$  [21]. The cases of  $0 < \delta < 1$  and  $1 < \delta < 2$  are the sub-Ohmic damping and super-Ohmic damping, respectively;  $\delta = 1$  is the Ohmic one, then  $\gamma(\omega)$  is equal to a constant and the noise is white. In Eq. (1),  $U(x)$  is considered to be a tilted periodic potential,

$$U(x) = -U_0 \cos\left(\frac{2\pi}{\lambda}x\right) - Fx. \quad (4)$$

The minima of  $U(x)$  vanish when the driving force  $F$  is taken to be the critical value:  $F_c = U_0 2\pi/\lambda = 1.0$ .

In the calculation, the natural unit ( $m = 1$  and  $k_B = 1$ ), the dimensionless parameters:  $U_0 = 1.0$ ,  $\lambda = 2\pi$ ,  $\gamma_\delta = 4.0$ , the smooth cutoff function  $f_c = \exp(-\omega/\omega_c)$  [21] with  $\omega_c = 4.0$ , and the time step  $\Delta t = 0.01$  are used. The test particles start from the origin of coordinate and have zero velocity, here  $2 \times 10^4$  test particles are used to describe the stochastic distribution of a Brownian particle.

### III. DIFFUSION AND MOBILITY

The quantities of foremost interest are the diffusion coefficient and the mobility. Here we generalize the both into non-Ohmic damping case with an arbitrary power index  $\delta$ ,

$$D^{(\delta)} := \lim_{t \rightarrow \infty} \frac{1}{\Gamma(1+\delta)} {}_0D_t^\delta \langle \Delta x^2(t) \rangle_\delta, \quad (5)$$

$$\mu_\delta := \lim_{t \rightarrow \infty} \frac{1}{F \sin(\frac{\delta\pi}{2})} {}_0D_t^\delta \langle x(t) \rangle_\delta, \quad (6)$$

where  ${}_0D_t^\delta$  denotes the fractional derivative. The algorithm for numerically calculating the two quantities is presented in the Appendix A.

#### A. Diffusion

We begin our studies from the situation of sub-Ohmic damping. We have noticed that the case of sub-diffusion dynamics has been discussed by Goychuk and Hänggi [22], where the GLE and the fractional Fokker-Planck equation approaches to the escape dynamics are used and compared. The escape is governed asymptotically by a power law whose exponent depends exponentially on both, the barrier height and the temperature. If the ratio of the barrier height to temperature is too large, the diffusion motion in a washboard potential well below a critical tilt cannot be observed numerically within a reasonable time window, i.e., nearly all the test particles are confined in the locking state. Therefore, we consider

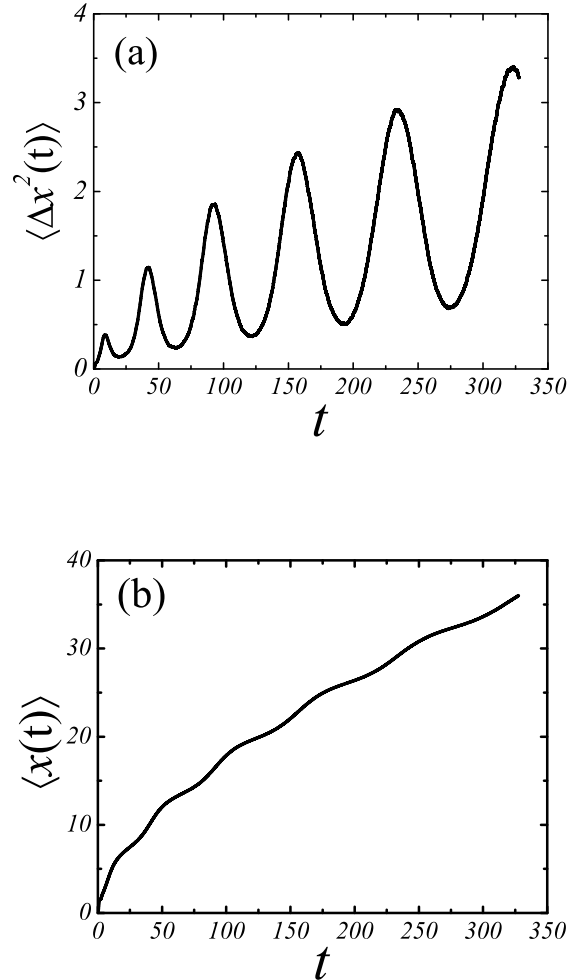


FIG. 1: (a) Time-dependent MSD of the sub-Ohmic damping particle of  $\delta = 0.6$  and (b) its mean displacement. The parameters used are:  $F = 5.0$  and  $T = 0.1$ .

the transport of sub-Ohmic damping particle subjected to a large driving force because of the efficiency of numerical simulation of GLE. Figures 1 (a) and 1 (b) show time-dependent MSD and MD of the particle of  $\delta = 0.6$ . With increasing the driving force  $F$  until local minima of the potential vanish, we find that the MSD of the particle shows a quasi-periodic oscillation. As long as the MSD of the particle experiences a quasi-period process, the particle will move the distance of a periodic length  $\Delta x = 2\pi$  along the direction of external force.

In Fig. 2, we plot the space probability distribution of a sub-Ohmic damping particle at different times. It is seen that with the evolution of time, the width of the probability distribution becomes periodically narrow when the particle moves in the bottom of a potential well; the one is broad when the particle arrives at the top of potential. Unlike the normal Ohmic damping par-

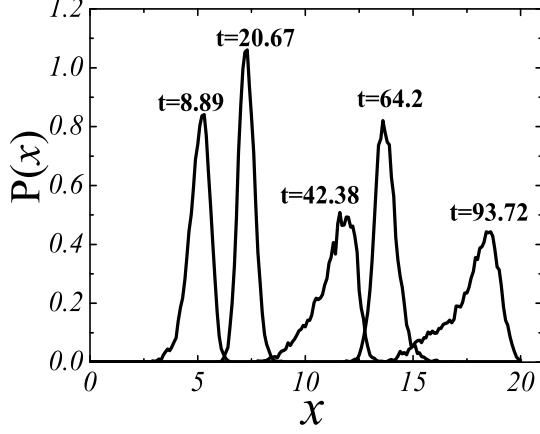


FIG. 2: The space probability distribution of the sub-Ohmic damping particle of  $\delta = 0.6$  at different times for  $F = 5.0$  and  $T = 0.1$ .

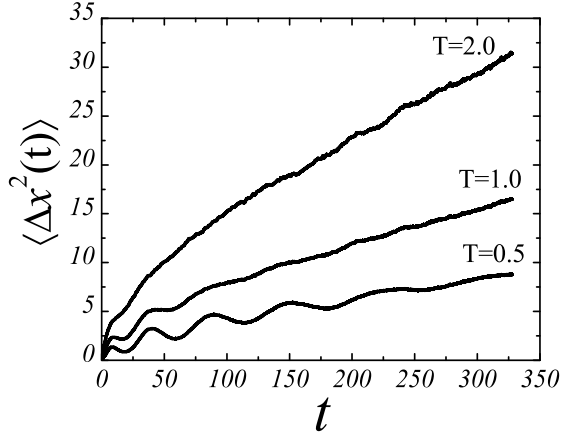


FIG. 3: The MSD of the particle with  $\delta = 1.7$  at a large driving force  $F = 5.0$  for various temperatures.

ticle [10], its probability distribution is not centralized. Our results can be interpreted as follows. Under the sub-Ohmic damping environment, the particle has a strong memory to its initial position and thus the diffusion in the coordinate space is much slow. If the potential have local minima, it is quite difficult for the particle to escape from the potential well, thus the particle is in a locking state during the period of simulation. As the local minima of the potential vanish, the particle subjected to a large driving force can enter the running state. Therefore, its distribution width is modulated by the periodic structure of potential and thus the MSD of particle occurs a quasi-periodic oscillation.

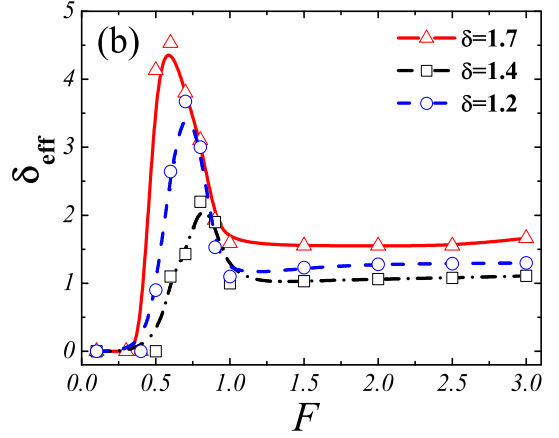
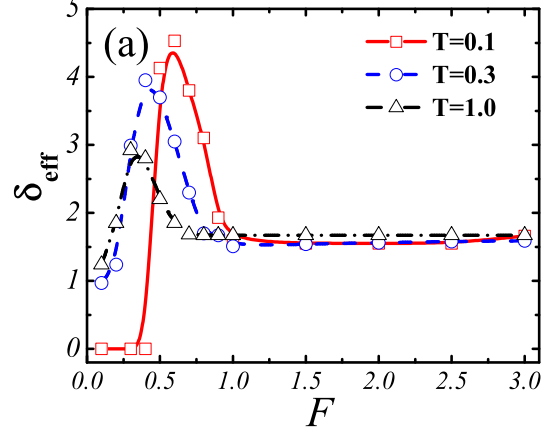


FIG. 4: (Color online) (a) The effective power index  $\delta_{\text{eff}}$  vs  $F$  for various  $T$  at  $\delta = 1.7$ . (b) The effective power index vs  $F$  for various  $\delta$  at  $T = 0.1$ .

Figure 3 shows that the quasi-periodic oscillation phenomenon becomes un conspicuous when the temperature increases for  $T > 1.0$  at  $F = 5.0$ . In this case the particle transfers completely into the running state and hence the structure of the potential might have less influence on the transport process.

In particular, in the case of super-Ohmic damping, we find numerically that the asymptotic MSD of the particle can be approximately written as a power function,

$$\langle \Delta x^2(t) \rangle = 2D_{\text{eff}}^{(\delta)} t^{\delta_{\text{eff}}(T, F)}, \quad (7)$$

where  $\delta_{\text{eff}}$  depends on  $T$  and  $F$ . Indeed, the index  $\delta_{\text{eff}}$  is not always equal to  $\delta$  like the Ohmic and the sub-Ohmic damping cases, but varies non-monotonically with  $F$ . For a moderate  $F$ , we find a prominent result: The effective power index  $\delta_{\text{eff}}$  exceeds 2 (i.e., the ballistic diffusion [23, 24, 25]) when the periodic potential is tilted observably

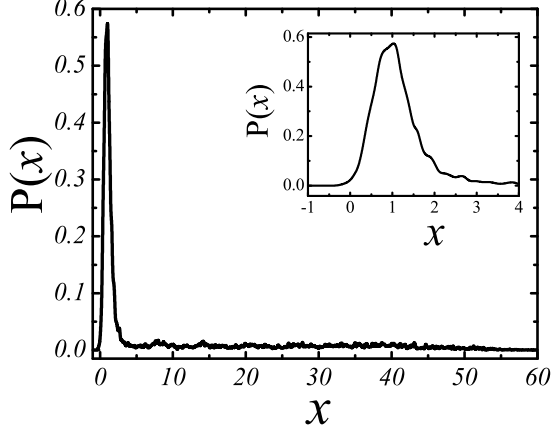


FIG. 5: The space probability distribution of the particle at time  $t = 50.0$ . The inset figure is the probability distribution at the locking state only. The parameters used are:  $\delta = 1.7$ ,  $F = 0.75$  and  $T = 0.1$ .

but its local minima still exist. Further analysis shows that the mysterious diffusion behavior is caused by the mixing of the locking state and the running state.

In Figs. 4 (a) and 4 (b), we plot the effective power index  $\delta_{\text{eff}}$  as a function of  $F$  for various  $T$  and  $\delta$ . It is seen from Fig. 4 (a) that the maximal value of  $\delta_{\text{eff}}$  versus  $F$  forward with increasing temperature. Similar behavior of  $\delta_{\text{eff}}$  for other super-Ohmic damping cases can also be observed, as shown in Fig. 4 (b). The smaller the value of  $\delta$  is, the larger  $F$  where the maximum  $\delta_{\text{eff}}$  appears. This can be interpreted qualitatively as follows. In the non-Markovian rate theory [26], for a sufficiently big ratio of barrier height to temperature, the super-diffusion in a periodic potential should turn out into the normal diffusion because the escape events are exponentially distributed in time and no overlong jumps can occur, so that  $\delta_{\text{eff}} = 1$  when the tilt of the potential is small. Also, for very large  $F$  and the structure effect of potential vanishing, all the test particles can move into a running state, thus  $\delta_{\text{eff}} = \delta$ . However, for a middle tilt, some test particles are confined in the potential well (in the locking state) and others drift quickly forward (in the running state). A proper proportion between the locking state and the running state should induce the maximum of the effective power index. Apparently it becomes easier for the particle in the locking state to escape the well and join the running state with the increase of either  $T$  or  $\delta$ . Therefore, the maximum of  $\delta_{\text{eff}}$  appears in the case of a small  $F$  at high temperature; in the case of large  $\delta$  at low temperature, respectively.

In Fig. 5, we illustrate the case of  $\delta = 1.7$  at the low temperature ( $T = 0.1$ ) and the middle tilt ( $F = 0.75$ ) to depict the coexistence of the two motion modes. The

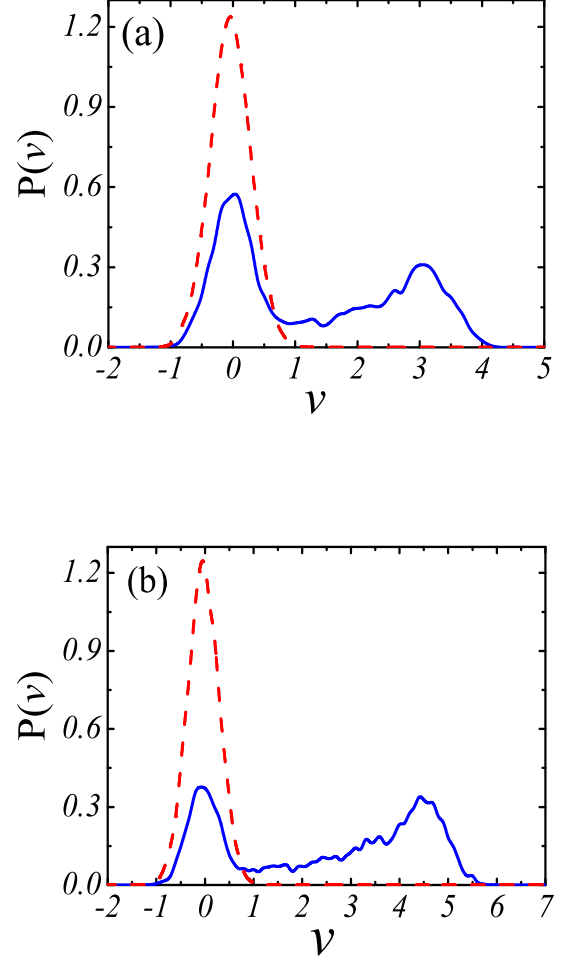


FIG. 6: (Color online) The velocity distributions of the particle at  $t = 150$  (a) and  $t = 330$  (b). The solid and dashed lines are the results of the super-Ohmic  $\delta = 1.7$  and Ohmic  $\delta = 1.0$  cases, respectively. The parameters used are:  $F = 0.75$  and  $T = 0.1$ .

backward and forward barrier heights are given by

$$U_1 = 2U_0 \sqrt{1 - \left(\frac{F\lambda}{2\pi U_0}\right)^2} + \frac{F\lambda}{\pi} \arcsin\left(\frac{F\lambda}{2\pi U_0}\right) + \frac{F\lambda}{2},$$

$$U_2 = 2U_0 \sqrt{1 - \left(\frac{F\lambda}{2\pi U_0}\right)^2} + \frac{F\lambda}{\pi} \arcsin\left(\frac{F\lambda}{2\pi U_0}\right) - \frac{F\lambda}{2}. \quad (8)$$

For a low temperature  $T < U_2 \ll U_1$ , the particle oscillates around the potential well, however, it still escapes over a barrier with a small probability. Since the barrier crossing process is quite slow at low temperature, the particle in the locking state has an approximate Gaussian space distribution centering at  $x_0$  [ $x_0 = \frac{\lambda}{2\pi} \arcsin(\frac{F\lambda}{2\pi U_0})$ ], as shown in the inset of Fig. 5. Once the test particle climbs over the barrier, it will no longer be restricted

again. Because the next hill of the titled periodic potential is lower than the present one, the kinetic energy of the particle gaining from the external driving force is greater than the dissipated energy due to the memory friction. The particle can enter the running state, so it drifts quickly along the direction of driving force. This results in a long tail appearing in the space probability distribution of the particle. For the Ohmic damping particle, once escaping over a barrier, it will slide to the next well and be trapped again. Hence the space probability distribution of the Ohmic damping particle is smashed into some small pieces and does not make up a running state.

Figures 6 (a) and 6 (b) show the coexistence of the two motion modes of the super-Ohmic damping particle, which can be more intuitively for the velocity distributions of the particle at times  $t = 150$  and  $t = 330$ . As expected, we do not find the coexistence phenomenon of two velocity modes appearing in the normal diffusion. It is seen that the super-Ohmic damping particle with an increasing probability enters the running state; the difference between the two center velocities in the running and rocking states becomes large with the evolution of time. This implies that as long as the test particles escape out of the well, they will be accelerated by the corrugated plane and join the running state. Of course, the present locking state occurs simply because we cannot observe the motion on the time-scale of our numerical simulation as the escape rate are very rare [22], but a new locking state relative to the forward running state should arise once the original locking state disappears.

### B. Mobility

The mobility determined by Eq. (6) as a function of the driving force is plotted in Fig. 7. Starting from zero tilt and switch adiabatically on the tilt  $F$ , the mobility of the particle remains zero when all the test particles are in the locking state until some of them join the running state. In comparison with the Ohmic damping case, we find that the hysteresis loop is broken and become stagger. At that point *A*, the mobility jumps to infinity and then drops to a constant at point *B* and keeps this constant with increasing  $F$  (point *C*). The point *B* corresponds to the critical force  $F = F_c$ , where the local minima of the corrugated plane vanish. When the driving force decreases adiabatically, all the test particles are kept in the running state and the mobility approaches a constant until the driving force becomes so small that most of the test particles are trapped in the potential wells. At the point *D*, the mobility falls to zero again. While for the usual Ohmic damping case shown in Fig. 7 (b), we find that a bistability occurs in the region  $0.7 < F < 1.4$ , which forms a hysteresis loop. However, for the sub-Ohmic damping case, we have not found similar hysteresis loop of mobility as what arises in the Ohmic damping case.

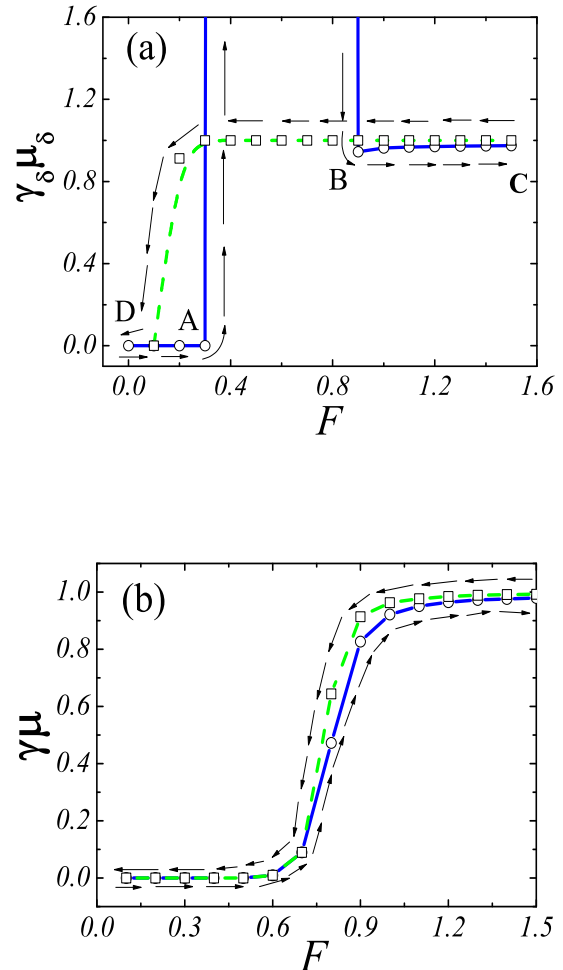


FIG. 7: (Color online) (a) The fractional mobility times the damping constant as a function of driving force in the super-Ohmic damping case with  $\delta = 1.7$ . (b) The normal Ohmic damping result with  $\gamma = 0.5$ . The solid and dashed lines correspond to the forward and backward processes, respectively. The temperature is  $T = 0.1$ .

### IV. SUMMARY

We have investigated the transport of a non-Ohmic damping particle in a titled periodic potential and reported a prominent finding: The titled periodic potential as a simple equipment which not only enhances the diffusion coefficient, but also changes diffusive behavior of the particle. This is due to a novel phenomenon of two motion modes: the locking state and the running state, which can appear and transfer in the corrugated plane. In the sub-Ohmic damping case, the mean square displacement of the particle shows a quasi-periodic property when the driving force is larger than the critical value where the minima of the potential vanish. While for

the super-Ohmic damping case, the two motion modes, namely, there exists two centers in the velocity distribution, can be coexisted and transferred. Thus the power index for the mean square displacement of the particle is enhanced. In comparison with the hysteresis loop of mobility of the normal case, the hysteresis loop of mobility of a super-Ohmic damping particle is broken.

The anomalous Brownian motion in a periodic potential is representable for many applications occurring in areas such as in condensed matter physics, chemical physics, molecular biology, communication theory, and so on. We are confident that both theoretical and experimental works in the future will help one to clarify further and shed more light onto all these intriguing issues and problems.

### ACKNOWLEDGEMENTS

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### APPENDIX A: NUMERICAL METHODS FOR FRACTIONAL CALCULUS

The so-called Riemann-Liouville fractional integral is defined through [27]

$${}_t I_t^{-\delta} f(t) = \frac{1}{\Gamma(\delta)} \int_{t_0}^t dt' \frac{f(t')}{(t-t')^{1-\delta}}, \quad t > 0, \delta > 0, \quad (\text{A1})$$

whereas its left-inverse  ${}_t D_t^\delta$  reads

$${}_t D_t^\delta := {}_t D_t^m {}_t I_t^{\delta-m}, \quad m-1 < \delta < m, m \in \mathbb{N}, \quad (\text{A2})$$

where  ${}_t D_t^m$  denotes the ordinary derivative of order  $m$ . In this present work, the case of  $t_0 = 0$  is concerned. For completeness, we define

$${}_0 I_t^0 = {}_0 D_t^0 = \mathbf{I}, \quad (\text{A3})$$

where  $\mathbf{I}$  is the identity operator. It is convenient to make use of the discrete operators of translation (shift) and finite differences to derive the approximative recursive expressions to the fractional differentiation operator  ${}_0 D_t^\delta$ . In a lucid way the theory of numerical differentiation and integration (with equidistant grid points) has been developed in Chapters 7 to 10 of Ref. [28]. See also Chapter 6 of Ref. [29].

Let  $\tau \in \mathbb{R}$ , we define the shifting operator  $E^\tau$  and the backward difference operator  $\nabla_\tau$  by them acting on a function  $u(t)$  for  $t \in \mathbb{R}$ ,

$$E^\tau u(t) = u(t + \tau), \quad (\text{A4})$$

$$\nabla_\tau u(t) = u(t) - u(t - \tau).$$

We furthermore have the relation, with  $\mathbf{I}$  as the identity operator,

$$\nabla_\tau = \mathbf{I} - E^{-\tau}. \quad (\text{A5})$$

Using these notations, we write the approximation  $[u(t) - u(t - \tau)]/h$  for the derivative  $u'(t)$  of a differentiable function  $u(t)$  as  $\nabla_h u(t)/h$  for a small positive  $h$  with accuracy  $O(h)$  as the function  $u(t)$  is sufficiently smooth. High order derivatives  $u^{(n)}(t) = {}_0 D_t^n u(t)$  ( $n \in \mathbb{N}$ ) with small  $h > 0$ , can be approximated by

$$[\nabla_h^{(n)} u(t)]/h^n = h^{-n} (\mathbf{I} - E^{-h})^n u(t), \quad (\text{A6})$$

again in case of  $u(t)$  being sufficiently smooth, with order of accuracy  $O(h)$ . The powers  $\nabla_h^{(n)}$  can be readily expanded via the binomial theorem

$$\nabla_h^{(n)} = \sum_{j=0}^n (-1)^j \binom{n}{j} E^{-jh}. \quad (\text{A7})$$

This leads to the known formula

$$h^{-n} \sum_{j=0}^n (-1)^j \binom{n}{j} u(t - jh) = {}_0 D_t^n u(t) + O(h). \quad (\text{A8})$$

The remarkable fact now is that these formulas can be generalized to the case of non-integer order of derivative. Replacing the positive integer  $n$  by a positive real number  $\delta$  amounts to use the formal power

$$\nabla_h^\delta = \sum_{j=0}^{\infty} (-1)^j \binom{\delta}{j} E^{-jh}, \quad (\text{A9})$$

in analogy to the expansion ( $E^{-h}$  replaced by the complex variable  $z$ )

$$(1 - z)^\delta = \sum_{j=0}^{\infty} (-1)^j \binom{\delta}{j} z^j, \quad (\text{A10})$$

which is convergent if  $|z| < 1$ . We thus get the Grünwald-Letnikov approximation:

$$h^{-\delta} \nabla_h^\delta u(t) = h^{-\delta} \sum_{j=0}^{\infty} (-1)^j \binom{\delta}{j} u(t - jh) = {}_0 D_t^\delta u(t) + O(h). \quad (\text{A11})$$

If  $u(t)$  decays to zero sufficiently fast as  $t \rightarrow \infty$ , in particular if  $u(t) = 0$  for  $t < 0$ ,  $\nabla_h^\delta$  will not diverge and hence for the latter case, we have

$$h^{-\delta} \nabla_h^\delta u(t) = h^{-\delta} \sum_{j=0}^{\lfloor t/h \rfloor} (-1)^j \binom{\delta}{j} u(t - jh). \quad (\text{A12})$$

By using the property of Gamma function

$$\Gamma(\delta)\Gamma(1-\delta) = \frac{\pi}{\sin(\pi\delta)},$$

we obtain the final recursion of calculating fractional calculus:

$${}_0 D_t^\delta u(kh) = \frac{h^{-\delta}}{\Gamma(-\delta)} \sum_{j=0}^{k-1} \frac{\Gamma(j-\delta)}{\Gamma(j+1)} u((k-j)h). \quad (\text{A13})$$

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